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A note on the resonant interaction between a surface wave and two interfacial waves

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Hill & Foda (1998) and Jamali (1998) have presented theoretical and experimental studies of the resonant interaction between a surface wave and two oblique interfacial waves. Despite many similarities between the findings there is one seemingly major difference. Hill & Foda's (1998) analysis indicated that there are only narrow bands of frequency, density ratio and direction angle within which growth is possible. On the other hand, Jamali (1998) predicted and observed wave growth over wide ranges of frequency and direction angle, and for all the density ratios that he investigated. We show that Hill & Foda's (1998) second-order representation of the dynamic interfacial boundary condition is missing a term proportional to the time derivative of the square of the velocity shear across the interface. When this missing term is included in the analysis, the resulting predictions are consistent with the laboratory experiments.

1. Introduction

Resuspension of material from the layers of fluid mud at the bottom of many lakes, estuaries and coastal waters, and from the unconsolidated sludge at the bottom of mine tailings ponds can be of significant practical importance (US Army Coastal Engineering Research Center 1984; Lawrence, Ward & MacKinnon 1991). One mechanism for resuspension is the generation of waves at the interface between the fluid mud (or mine tailings) and the overlying water due to surface wave motion. Hill (1997) and Jamali (1998) have studied this problem independently. An account of the work of Hill (1997) is given in Hill & Foda (1998). The present paper provides an overview of part of Jamali's (1998) work. Both studies adopted essentially the same approach to analyse the interaction between a surface wave and two oblique interfacial waves. They found that the interfacial waves are short, have a frequency of nearly half that of the surface wave, and propagate in nearly opposite directions. In laboratory tanks, these interfacial waves reflect from the sidewalls to form a three-dimensional standing wave pattern. Hill & Foda (1998) excited the lowest mode in the cross-tank direction, whereas Jamali (1998) excited higher modes.

Despite many similarities between the findings of Hill & Foda (1998) and Jamali (1998) there is one seemingly major difference. Hill & Foda's (1998) analysis predicts only narrow bands of frequency, density ratio and direction angle within which growth is possible, leading them to state: 'The net effect of these various bounds is that instability of the internal waves, i.e. internal wave growth, is found to be a very selective process, occurring under very specific conditions.' On the other hand,

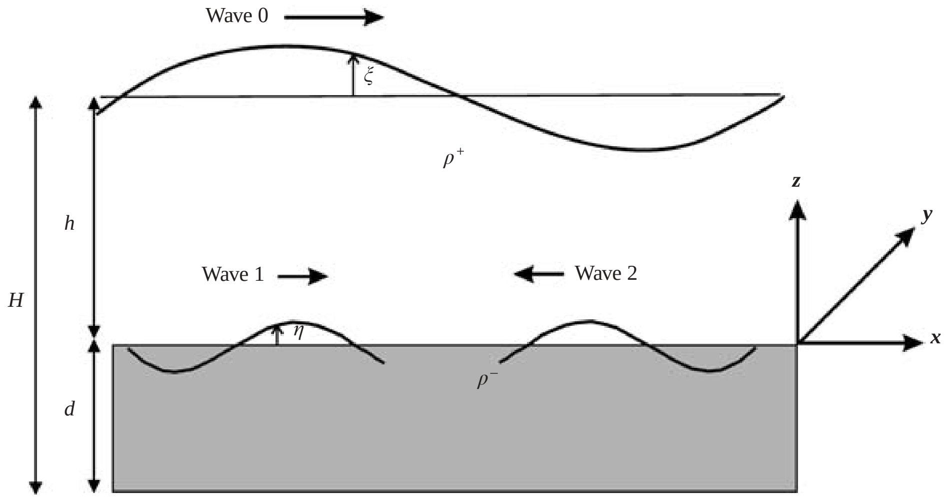


FIGURE 1. Definition diagram for the interaction of a surface wave with two interfacial waves.

Jamali (1998) predicted and observed wave growth over wide ranges of frequency and direction angle, and for all the density ratios he investigated. The purpose of the present study is to investigate these contradictory results.

2. Theoretical background

We now present the basic theoretical development common to both Hill & Foda (1998) and Jamali (1998). Consider the inviscid, two-layer fluid system shown in figure 1. The system is assumed to be three-dimensional and horizontally infinite. The coordinate system xyz is located on the interface. The depth of the upper layer is h, the lower layer d and the total depth H. The densities of the upper and lower layers are ρ^+ and ρ^-, respectively. Although viscosity is important in damping the waves, it is not central to our primary objective of comparing the results of the inviscid studies of Jamali (1998) and Hill & Foda (1998). Therefore, viscosity will not be treated in the present analysis. The resonant triad consists of a surface wave and two interfacial waves. Without loss of generality, the surface wave is assumed to travel in the positive x-direction and the two interfacial waves in the (x, y)-plane. The interfacial waves 1 and 2 have directional angles θ_1 and θ_2 with respect to the surface wave. The three waves satisfy the resonance conditions:

$$\boldsymbol{k}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2, \quad \omega_0 = \omega_1 + \omega_2, \tag{1}$$

where the wave frequencies ω_i have real positive values and the wavenumber vectors $\boldsymbol{k}_i = (k_{ix}, k_{iy})$. The subscript 0 denotes the surface wave and the subscripts 1 and 2 the two interfacial waves. The dispersion relationship for this system is well known (see Lamb 1932, art. 231). In many circumstances $\delta = 1 - r \ll 1$, where $r = \rho^+/\rho^-$. From the simultaneous solution of (1) and the dispersion relations, $k_i = |\boldsymbol{k}_i|$, ω_1, ω_2 and θ_2 are approximated by:

$$k_1 \sim k_2 = \frac{k_0 \tanh(k_0 H)}{2\delta} + O(1), \quad \omega_1 \sim \omega_2 = \tfrac{1}{2}\omega_0 + O(\delta), \quad \text{and } \theta_2 = \theta_1 + \pi + O(\delta). \tag{2}$$

These results are consistent with the observations of both Hill & Foda (1998) and Jamali (1998).

With the assumptions of incompressibility and irrotational flow in each of the layers, the fluid motion can be described by velocity potentials $\phi^{\pm}(x, y, z, t)$ in the upper and the lower layers, respectively. These potentials satisfy Laplace's equation, and are subject to boundary conditions on the free surface, the interface between the layers, and the solid bed. Of particular relevance to the present study are the boundary conditions on the interface $z = \eta(x, y, t)$:

$$\eta_t + \phi_x^{\pm}\eta_x + \phi_y^{\pm}\eta_y = \phi_z^{\pm}, \tag{3a}$$

$$\rho^+[\phi_t^+ + ((\phi_x^+)^2 + (\phi_y^+)^2 + (\phi_z^+)^2)/2 + gz] = \rho^-[\phi_t^- + ((\phi_x^-)^2 + (\phi_y^-)^2 + (\phi_z^-)^2)/2 + gz], \tag{3b}$$

where $\eta(x, y, t)$ is the displacement of the interface.

The standard procedure for a weakly nonlinear interaction analysis (Craik 1985) yields:

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \mathrm{i}\alpha_1 a_2^* A, \quad \frac{\mathrm{d}a_2}{\mathrm{d}t} = \mathrm{i}\alpha_2 a_1^* A, \tag{4}$$

where $a_1(t)$ and $a_2(t)$ are the complex amplitudes of the interfacial waves, A is the complex amplitude of the surface wave, and the interaction coefficients α_1 and α_2 are real, but complicated functions of the characteristic parameters of the problem. In both Hill & Foda (1998) and Jamali (1998), the surface wave has much more energy than the interfacial waves and A is assumed constant. The interfacial wave amplitudes are then given by $a_1, a_2 \propto \exp(\pm\alpha|A|t)$, where $\alpha = \sqrt{\alpha_1\alpha_2}$. For the case where $\delta \ll 1$, then Jamali, Seymour & Lawrence (2003) have shown that:

$$\alpha = \frac{\omega_0^3}{4g\sinh^2(k_0H)}[\cosh(k_0H)\cosh(k_0d)\sin^2\theta_1 + \cosh(k_0h)] + O(\delta), \tag{5}$$

which is positive for all values of ω_0 and θ_1, in contrast to the predictions of Hill & Foda (1998). Jamali (1998) gives general expressions for the interaction coefficients when the relative density difference is not restricted to be small; Hill & Foda (1998) do not, but even if they had, the expressions are so long that it would be extremely difficult to make a direct comparison. Fortunately, a direct comparison of interaction coefficients may not be necessary to resolve the difference between the two studies.

3. Theoretical comparison

A potential source of discrepancy between the two studies is in the treatment of the interfacial boundary conditions. After applying Taylor series expansions about $z = 0$, (3) become:

$$\phi_z^+ - \eta_t = -\phi_{zz}^+\eta + \phi_x^+\eta_x + \phi_y^+\eta_y + O(\varepsilon^3), \tag{6}$$

$$\phi_z^- - \eta_t = -\phi_{zz}^-\eta + \phi_x^-\eta_x + \phi_y^-\eta_y + O(\varepsilon^3), \tag{7}$$

$$\Phi_t + g'\eta = \Lambda - \eta\Phi_{tz} + O(\varepsilon^3), \tag{8}$$

where $\Phi = \phi^- - r\phi^+$, $g' = \delta g$, $\Lambda = (r|\nabla\phi^+|^2 - |\nabla\phi^-|^2)/2$, and ε is the wave steepness. Multiplying (7) by g, and subtracting (6) multiplied by rg yields:

$$g\Phi_z - g'\eta_t = -g\eta\Phi_{zz} + g\eta_x\Phi_x + g\eta_y\Phi_y + O(\varepsilon^3). \tag{9}$$

Adding the time derivative of (8) to (9), the dynamic interfacial boundary condition becomes:

$$\Phi_{tt} + g\Phi_z = \Lambda_t - \eta_t \Phi_{tz} + \frac{\Phi_t}{g'}\{\Phi_{ttz} + g\Phi_{zz}\} + g\eta_x \Phi_x + g\eta_y \Phi_y + O(\varepsilon^3), \tag{10}$$

knowing that from (8) $\eta = -\Phi_t/g' + O(\varepsilon^2)$. The corresponding result given by Hill (1997, equation 3.11) is:

$$\Phi_{tt} + g\Phi_z = 2\Lambda_t + \frac{\Phi_t}{g'}\{\Phi_{ttz} + g\Phi_{zz}\} + O(\varepsilon^3). \tag{11}$$

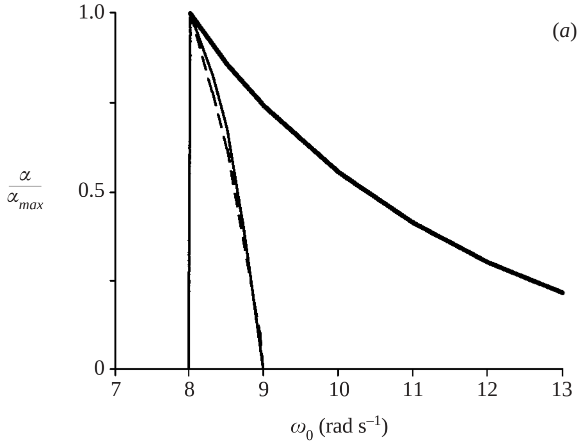
To determine whether or not (10) and (11) are equivalent, we define S to be equal to the right-hand side of (10) minus the right-hand side of (11), to obtain:

$$S = -\Lambda_t - \eta_t \Phi_{tz} + g\eta_x \Phi_x + g\eta_y \Phi_y + O(\varepsilon^3) = \frac{-r}{2(1-r)}\{(\phi_x^+ - \phi_x^-)^2 + (\phi_y^+ - \phi_y^-)^2_t\} + O(\varepsilon^3). \tag{12}$$

Thus, Hill's (1997) expression for the dynamic interfacial boundary condition is missing the non-zero term S, which is proportional to the time derivative of the square of the velocity shear across the interface. Given that a correct representation of the interfacial boundary condition is crucial to an accurate determination of growth rates, it is important that we examine the consequences of the above result.

We have compared growth rate predictions of Hill & Foda (1998, figure 7*b*) with those computed using the solution of Jamali (1998), first with (10) and then with (11) as the interfacial boundary condition. The results of the three analyses are plotted in figure 2(*a*). These are for the experimental parameters $h = 12.5$ cm, $d = 7.9$ cm, $r = 0.85$, $k_{1y} = k_{2y} = 20.6\ \mathrm{m}^{-1}$, and $\omega_0 = 8.7\ \mathrm{s}^{-1}$ (corresponding to the second set of experiments of Hill & Foda 1998), which yield $k_0 = 8.3\ \mathrm{m}^{-1}$, $k_1 = 26.6\ \mathrm{m}^{-1}$, $k_2 = 22.3\ \mathrm{m}^{-1}$, $\theta_1 = 50.7^\circ$, and $\theta_2 = 247^\circ$. All three analyses correctly predict the lower bound on the surface wave frequency, which is a purely kinematic constraint. The calculations based on the interfacial boundary condition (11), i.e. without the term S, provide a close match to those of Hill & Foda (1998). However, when S is included, the predicted growth rates are generally higher, and there is no upper bound on surface wave frequency. Thus, it seems that an important difference between the two studies is that Hill & Foda (1998) use a dynamic interfacial boundary condition that omits the velocity shear at the interface.

As a consequence of using an incorrect boundary condition, Hill & Foda (1998) predict an upper bound on the surface wave frequency, corresponding to a threshold where the interaction coefficients change from having the same sign to having different signs, resulting in an imaginary α. However, when S is included, there is no such threshold on ω_0. In fact, for interaction coefficients in a three-wave interaction $\alpha_1/\omega_1 = \alpha_2/\omega_2$, see Simmons (1969) and Phillips (1977). Since ω_1 and ω_2 are positive, the interaction coefficients must always be of the same sign. Furthermore, Hill & Foda's (1998) result that the interaction coefficients may have different signs contradicts the analysis of Hasselmann (1967) who states: 'The non-linear coupling between two infinitesimal components 1 and 2 and a finite component 0 whose wave-numbers and frequencies satisfy the resonant conditions $\boldsymbol{k}_0 = \boldsymbol{k}_1 \pm \boldsymbol{k}_2$, $\omega_0 = \omega_1 \pm \omega_2$, is unstable for the sum interaction and neutrally stable for the difference interaction.' Hasselmann showed that these results apply for all conservative coupled-mode systems independent of the details of the coupling.



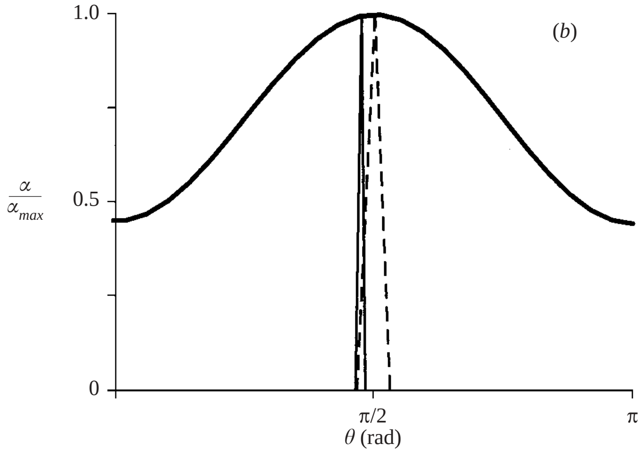


FIGURE 2. Predictions of the variation of growth rate with (*a*) surface wave frequency, and (*b*) direction angle of the interfacial wave 1; $h=12.5\,\mathrm{cm}$, $d=7.9\,\mathrm{cm}$, $r=0.85$, $k_{1y}=k_{2y}=20.6\,\mathrm{m}^{-1}$, $\omega_0=8.7\,\mathrm{s}^{-1}$ (corresponding to the second set of experiments of Hill & Foda 1998). ━━, predictions obtained using the solution of Jamali (1998) with boundary condition (10) and - - -, with boundary condition (11). —, predictions presented by Hill & Foda (1998, figure 7*b*). The predicted growth rates are scaled by $\alpha_{\max}$, the maximum value predicted by each of the analyses, because Hill & Foda's graph (1998, figure 7*b*) is subject to a scaling error (Hill personal communication).

Hill & Foda (1998) also examined the effect of interfacial wave direction on α. Their results are plotted in figure 2(*b*) with those obtained using Jamali's (1998) equations, first with (10), and then with (11) as the interfacial boundary condition. The predictions of Hill & Foda (1998), and the predictions obtained using (11), show only a narrow band at $\theta_1 \approx \pm\pi/2$ within which growth is possible. The width of this narrow band is not the same in each case, for reasons that we are not aware of. However, when Jamali's (1998) complete equations are used, α is always positive with a maximum at $\theta_1 \approx \pm\pi/2$, so the response does not appear to be narrow banded. Also, the experimental data presented in Hill & Foda (1998) indicate direction angles substantially different from $\pm\pi/2$. To further test the above results, we will examine some of the experimental observations of Jamali (1998).

Exp.	H (cm)	d (cm)	d/H	T_0 (s)	ω_0 (s^{-1})	$\bar{l}_x$ (cm)	$\bar{k}$ (m^{-1})	m	$\bar{\theta}$ (deg)
1	*15.8*	*3.8*	0.24	*1.30*	4.8	*20*	0.43	*2*	43
2	*16.5*	*3.8*	0.23	*1.19*	5.3	*67*	0.46	*3*	78
3	*16.3*	*3.8*	0.23	*1.11*	5.7	*27*	0.50	*3*	62
4	*16.4*	*3.9*	0.24	*0.96*	6.5	*17*	0.70	*4*	58
5	*16.0*	*3.9*	0.24	*0.90*	7.0	*21*	0.80	*5*	68
6	*16.5*	*3.9*	0.24	*0.84*	7.5	*40*	0.91	*6*	80
7	*16.5*	*3.9*	0.24	*0.80*	7.9	*16*	0.98	*6*	66

TABLE 1. Summary of the experimental parameters. Measured values are given in italics. The density relative difference $\delta = 0.04$ in each experiment.

4. Experimental investigation

The experiments of Jamali (1998) were performed in a wave tank 4 m long, 21 cm wide and 35 cm deep. A video camera was used to record the experiments and to obtain the measurements of heights, frequencies and wavelengths of the surface and the interfacial waves. A series of seven experiments was performed, see table 1. The experimental strategy was to vary the surface wave period, T_0, while holding the other controllable parameters approximately constant; i.e. $\delta = 0.04$ and $d/H = 0.24$. In each experiment, the along channel wavelength $\bar{l}_x$ was measured. The measurements were made with an experimental error of ± 1 mm in H and d, ± 0.02 s in T_0, ± 0.002 in δ, and ± 1 cm in $\bar{l}_x$.

A brief discussion of the experimental results will be presented here with the objective of providing enough information to make comparisons with the work of Hill & Foda (1998). For a more detailed account see Jamali (1998). In each of the experiments, a three-dimensional standing wave pattern was observed at the interface. By recognizing that the standing wave pattern is generated by the superposition of the pair of the interfacial waves with their reflections from the sidewalls, Jamali (1998) has shown that the interfacial displacement:

$$\eta_{\text{int}}(x, y, t) = b(x, t)\cos(\bar{k}_x x)\cos(\bar{k}_y y)\sin\left(\tfrac{1}{2}\omega_o t\right), \tag{13}$$

where the standing wave amplitude $b(x, t)$ is a slowly varying function of x and t. The standing wave pattern is characterized by the wavenumber vector $\bar{\boldsymbol{k}} = (\bar{k}_x, \bar{k}_y)$, where

$$\bar{k}_x = (k_{1x} - k_{2x})/2, \quad \bar{k}_y = m\pi/B. \tag{14}$$

B is the width of the channel, and the mode number m is the number of half wavelengths in the cross-tank direction. In a tank, $\bar{k}_y = k_{1y} = k_{2y}$. Flow with $m = 4$ (exp. 4) is shown in figure 3. The interfacial waves are short compared to the surface wave in accordance with (2) and (14).

The direction of the waves in the standing interfacial pattern can be defined by angle $\bar{\theta} = \arcsin(\bar{k}_y/\bar{k})$. Given that $\delta \ll 1$ in the experiments, then from (2) $k_{1x} = -k_{2x} + O(1)$ and $\bar{k} = \bar{k}_1 + O(1)$, which gives $\bar{\theta} = \theta_1 + O(\delta)$. The asymptotic solution (5) states that the growth rate increases as the sine of the direction angle increases. Therefore, for a given frequency, the interfacial waves appear at the largest m satisfying $\sin(\bar{\theta}) = m\pi/(B\bar{k}) \leqslant 1$, or:

$$m = \text{INT}(\bar{k}B/\pi). \tag{15}$$

From table 1, we see that as the surface wave frequency increases, $\bar{k}$ increases, resulting in either a decreasing $\bar{\theta}$ at the same m; or, if the increase in $\bar{k}$ is large enough, an

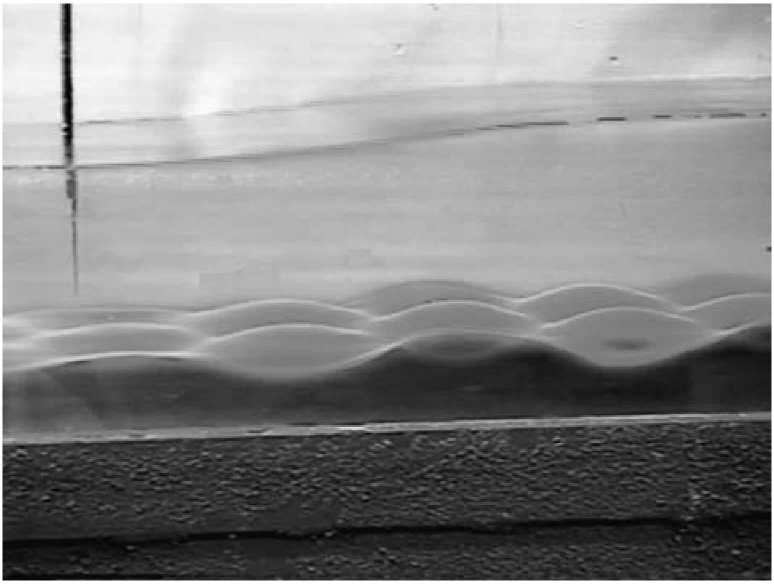


FIGURE 3. Observed three-dimensional standing interfacial wave in experiment 4, $m = 4$.

increase in m. Increasing the surface wave frequency from 4.8 s^{-1} (exp. 1) to 7.9 s^{-1} (exp. 7) results in an increase in m from 2 to 6. The direction angle varied between 43° and 80°.

5. Discussion

Predicted growth rates were obtained using the solution to the evolution equations of Jamali (1998), first with (10), and then with (11) as the interfacial boundary condition. The predicted growth rates are plotted as a function of $\bar{k}B/\pi$ in figure 4 along with the predicted and measured mode numbers. The bands of instability are very narrow when Hill & Foda's (1998) boundary condition is used, and only two of the experiments fall within these narrow bands. However, when the correct boundary condition is included, the predicted growth rate is positive for all $\bar{k}B/\pi$, consistent with the fact that instability was observed in all seven experiments. Also, the observed mode numbers are correctly predicted by (15), which is a confirmation of the result that the more oblique the interfacial waves, the higher their growth rate, see (5).

Besides experiments with $\delta = 0.04$, Jamali (1998) also conducted a series of experiments with $\delta = 0.07$, 0.11 and 0.14. In all cases, instability was observed, the only caveat being that the surface wave amplitude be greater than a critical value necessary to overcome the effects of viscosity at the interface, the walls and the bed of the tank. The observation that viscosity can suppress the instability is in agreement with the theoretical findings of Jamali (1998), Hill & Foda (1999) and Jamali *et al.* (2003). Thus, if viscosity had been included in Hill & Foda's (1998) analysis it would have restricted the conditions necessary for instability even further.

Thus, the main consequence of Hill & Foda (1998) using (11) as the interfacial boundary condition is the prediction that there exist only narrow bands of frequency, density ratio and direction angle within which growth is possible. Whereas the experiments of Jamali (1998) exhibited instability at all values of frequency, density

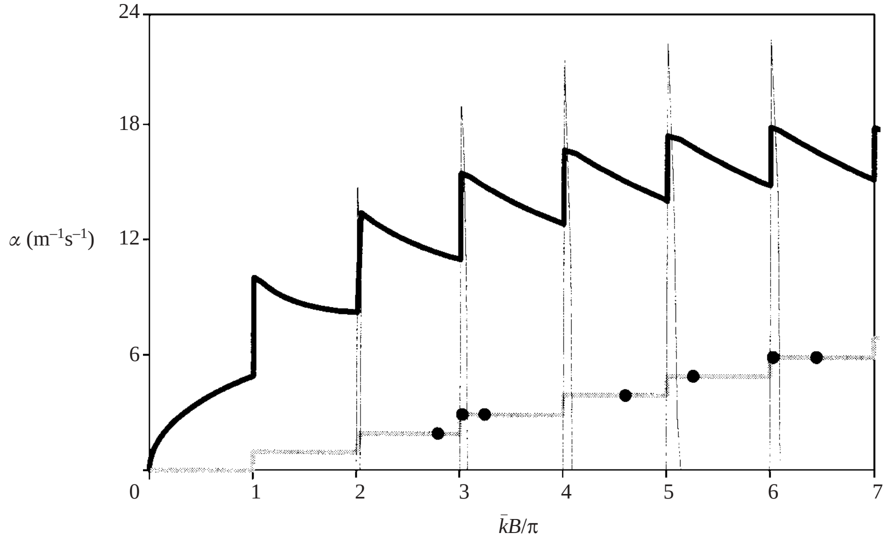


FIGURE 4. —, predictions of growth rate made using the solution of Jamali (1998) with boundary condition (10) and —, boundary condition (11) as a function of $\bar{k}B/\pi$ for experiments 1–7; $\delta = 0.04$, $d = 3.9$ cm, $H = 16.3$ cm. Also plotted are —, the predicted and ●, observed mode numbers.

ratio and direction angle tested, consistent with predictions made using (10) as the interfacial boundary condition.

6. Summary and conclusions

Two studies of the interaction between a surface wave and two interfacial waves have produced conflicting results. Hill & Foda (1998) claimed that the instability of the interfacial waves is a very selective process, occurring under very specific conditions, but Jamali (1998) predicted and observed wave growth over a wide range of parameters. The crucial difference between the two studies is in the dynamic interfacial boundary condition. The boundary condition used by Hill & Foda (1998) is missing a term that is proportional to the time derivative of the square of the velocity shear across the interface. If this term is included in the analysis, the theoretical predictions match the results of Jamali's (1998) laboratory experiments.

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REFERENCES

CRAIK, A. D. D. 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

HASSELMANN, K. 1967 A criterion for nonlinear wave stability. *J. Fluid Mech.* **30**, 737–739.

HILL, D. F. 1997 The subharmonic resonance of interfacial waves by progressive surface waves. PhD thesis, University of California at Berkeley.

HILL, D. F. & FODA, M. A. 1998 Subharmonic resonance of oblique interfacial waves by a progressive surface wave. *Proc. R. Soc. Lond.* A **454**, 1129–1144.

Hill, D. F. & Foda, M. A. 1999 Effects of viscosity and elasticity on the nonlinear resonance of internal waves. *J. Geophys. (Oceans)* **104** (C5), 10 951–10 957.

Jamali, M. 1998 Surface wave interaction with oblique internal waves. PhD thesis, University of British Columbia, Canada.

Jamali, M., Seymour, B. & Lawrence, G. 2003 Asymptotic solution of a surface–interfacial wave interaction. *Phys. Fluids* **15**, 47–55.

Lamb, H. 1932 *Hydrodynamics*. Cambridge University Press.

Lawrence, G. A., Ward, P. R. B. & MacKinnon, M. D. 1991 Wind-wave-induced suspension of mine tailings in disposal ponds – a case study. *Can. J. Civil Engng* **18**, 1047–1053.

Phillips, O. M. 1977 *The Dynamics of the Upper Ocean*. Cambridge University Press.

Simmons, W. F. 1969 A variational method for weak resonant wave interactions. *Proc. R. Soc. Lond.* A **309**, 551–575.

U. S. Army Coastal Engineering Research Center 1984 *Shore Protection Manual*. Vicksburg, MS.